QUANTITATIVE STABILITY OF SOBOLEV INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. We study quantitative stability results for different classes of Sobolev inequalities on general compact Riemannian manifolds. We prove that, up to constants depending on the manifold, a function that nearly saturates a critical Sobolev inequality is quantitatively $W^{1,2}$-close to a non-empty set of extremal functions, provided that the corresponding optimal Sobolev constant satisfies a suitable strict bound. The case of sub-critical Sobolev inequalities is also covered. Finally, we discuss degenerate phenomena in our quantitative controls.

CONTENTS

1. INTRODUCTION

In the standard round sphere $\mathbb{S}^d$, for $d > 2$, the sharp Sobolev inequality reads:

$$(1.1) \qquad \|u\|_{L^{2^*}}^2 \leq \mathsf{S}_d^2 \|\nabla u\|_{L^2}^2 + \mathrm{Vol}(\mathbb{S}^d)^{-2/d} \|u\|_{L^2}^2, \qquad \forall u \in W^{1,2}(\mathbb{S}^d),$$

where $2^* := 2d/(d-2)$ is the Sobolev conjugate exponent, and $\mathsf{S}_d > 0$ is the optimal constant in the sharp Euclidean Sobolev inequality

$$(1.2) \qquad \|u\|_{L^{2^*}} \leq \mathsf{S}_d \|\nabla u\|_{L^2}, \qquad \forall u \in \dot{W}^{1,2}(\mathbb{R}^d).$$

The value

$$\mathsf{S}_d = \left(\frac{2^* - 2}{d} \right)^{\frac{1}{2}} \mathrm{Vol}(\mathbb{S}^d)^{-1/d},$$

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was found by Aubin [Aub76b] and Talenti [Tal76], while the validity of (1.1) goes back to [Aub76a] where also non-constant *extremal functions* were classified (see [Heb99, Chapter 5]):

$$(1.3) \qquad u_{a,b,z_0} := a(1 - b\cos(d(\cdot, z_0)))^{\frac{2-d}{2}}, \qquad \text{with } a \in \mathbb{R}, \ b \in (0,1), \ z_0 \in \mathbb{S}^d.$$

Here d is the geodesic distance on $\mathbb{S}^d$. In light of this, a natural question is the one of stability:

<p align="center">If $u \neq cst$ nearly *saturates* (1.1), is u close to some u_{a,b,z_0}?</p>

This question is equivalent (up to a change of coordinates via the stereographic projection, see e.g. [FL12, Appendix A]) to the analogous question raised in [BL85] for the Euclidean Sobolev inequality (1.2) and was addressed in Bianchi and Egnell's work [BE91]. The *quantitative stability* for (1.2) initiated in *loc. cit.* has been then studied thoroughly in subsequent works [CFMP09, FN19, Neu20, FZ22] dealing with p-Sobolev inequalities for $p \neq 2$, and recently in [DEF$^+$23, DEF$^+$24] with explicit constant (see also to the related [Kö23] and [Fra23]).

Aim of this manuscript is to study quantitative stability of optimal Sobolev inequalities on a general closed Riemannian manifold in connection with the existence of extremal functions. We now start by recalling the notions of optimal constants appearing in the celebrated AB-program referring to [Heb99], see also [Aub98, DH02a] and references therein. Given $d > 2$ and (M, g) a closed, i.e. compact and boundaryless, d-dimensional smooth Riemannian manifolds, we can consider for constants $A, B \geq 0$ the following type of Sobolev inequalities

$$(\ast) \qquad \|u\|_{L^{2^*}}^2 \leq A\|\nabla u\|_{L^2}^2 + B\|u\|_{L^2}^2, \qquad \forall u \in W^{1,2}(M).$$

Notice that an equation as (1.2) cannot hold due to the presence of constant functions and, differently from the Euclidean case, it is not straightforward to consider a single sharp Sobolev inequality due to the presence of two constants. The AB-program then starts with the definition of the following Sobolev constants:

$$\alpha(M) := \inf\{A \ : \ (\ast) \text{ holds for some } B\}, \qquad \beta(M) := \inf\{B \ : \ (\ast) \text{ holds for some } A\},$$

where we understand the infimum over an empty set as being $+\infty$. The first natural problem is then to determine the value of $\alpha(M)$ and $\beta(M)$. It is rather straightforward to see (e.g. [Heb99, Sec 4.1]) that the latter satisfies

$$\beta(M) = \mathrm{Vol}_g(M)^{-2/d}.$$

On the other hand, the value of $\alpha(M)$ turns out to be linked to the sharp Euclidean Sobolev constant. More precisely, in [Aub76b] (see also [Heb99, Theorem 4.5]) it is shown that

$$(1.4) \qquad\qquad\qquad\qquad \alpha(M) = \mathsf{S}_d^2.$$

In particular, $\alpha(M)$ does not depend on M but only on its dimension d. A more subtle question is whether these two constants are actually attained, i.e. if they are minima. For instance, for $M = \mathbb{S}^d$ we have in (1.1) the validity of a Sobolev inequality with $A = \alpha(\mathbb{S}^d)$ and $B = \beta(\mathbb{S}^d)$ and they are attained by the family (1.3).

More generally, and starting from the easiest one, it was shown in [Bak94] that $\beta(M)$ is a minimum (see also [Heb99, Theorem 4.2]), i.e. there is $\overline{A} > 0$ so that $(\ast)$ holds with $\overline{A}$ and $\beta(M)$. It is instead a deep result of [HV96] that $\alpha(M)$ is attained, i.e. there is $\overline{B} > 0$ so that $(\ast)$ holds with $\alpha(M)$ and $\overline{B}$. Thanks to these results, we can define two further notions of Sobolev constants. We can fix $B = \beta(M)$ and proceed with the A-part of this program, i.e. minimizing over the admissible $A > 0$ and consider more generally all sub-critical ranges

$$(\ast_A) \qquad\qquad\qquad \|u\|_{L^q}^2 \leq A\|\nabla u\|_{L^2}^2 + \mathrm{Vol}_g(M)^{2/q-1}\|u\|_{L^2}^2,$$

for $q \in (2, 2^*]$. Conversely, we can fix $A = \alpha(M)$ and proceed with the B-part of this program, i.e. minimizing over the admissible $B > 0$ and consider

$(*_B)$ $$\|u\|_{L^{2^*}}^2 \leq \mathsf{S}_d^2 \|\nabla u\|_{L^2}^2 + B\|u\|_{L^2}^2.$$

Then, we can define the corresponding notions of second-best optimal constants

$$A_q^{\mathrm{opt}}(M) := \inf\{A > 0 \colon (*_A) \text{ holds with } A\}, \quad B_{2^*}^{\mathrm{opt}}(M) := \inf\{B > 0 \colon (*_B) \text{ holds with } B\},$$

Differently from $\alpha(M), \beta(M)$ these two constants are automatically minima. Moreover, universal bounds depending on the geometry of M are given for $A_q^{\mathrm{opt}}(M)$ in [DH02b, Theorem 4.4] (see also [NV22, Proposition 5.1] for general q), and for $B_{2^*}^{\mathrm{opt}}(M)$ in [HV95].

Having established two classes of optimal Sobolev inequalities, it is natural to investigate the existence of extremal functions. Let us consider the following sets:

$$\mathcal{M}_q(A) := \{u \in W^{1,2} \setminus \{0\} \colon \text{ equality holds with } u \text{ in } (*_A) \text{ for } A = A_q^{\mathrm{opt}}(M)\},$$
$$\mathcal{M}_{2^*}(B) := \{u \in W^{1,2} \setminus \{0\} \colon \text{ equality holds with } u \text{ in } (*_B) \text{ for } B = B_{2^*}^{\mathrm{opt}}(M)\}.$$

The existence and compactness properties of extremal functions become challenging in the critical case due to the apriori lack of compactness of the embedding $W^{1,2} \hookrightarrow L^{2^*}$. We will see that compactness properties are related to the values of the constants $A_{2^*}^{\mathrm{opt}}(M), B_{2^*}^{\mathrm{opt}}(M)$. Notice however that $\mathcal{M}_q(A)$ always contains constant functions thanks to the value of the B-constant in $(*_A)$. In particular, it is never empty as defined. However, we do not restrict our analysis to *non-constant* extremal functions in this case, as these could be the only extremal functions (see [Fra22] and below). Finally, we refer the reader to [HV01, DD01, BM07, BM09, Bar10, BM12], where extremal functions in the AB-program have been investigated.

Statement of the main results. Here we present our main quantitative stability results. As already observed, $\mathcal{M}_q(A)$ is never empty as it contains constant functions. Also, by definition of $\alpha(M)$, we always have $A_{2^*}^{\mathrm{opt}}(M) \geq \mathsf{S}_d^2$. However, assuming this inequality to be strict, we gain pre-compactness of (normalized) extremizing sequences via concentration-compactness methods in the spirit of [Lio84, Lio85], cf. Proposition 2.1. Under this assumption, or more easily in the sub-critical range, we can prove the following.

Theorem 1.1. *Let (M, g) be a closed d-dimensional Riemannian manifold, $d > 2$. Assume that either $2 < q < 2^*$, or $q = 2^*$ and $A_{2^*}^{\mathrm{opt}}(M) > \mathsf{S}_d^2$. Then, there are non-negative constants C, γ depending on (M, g) and on q so that, for every $u \in W^{1,2}(M) \setminus \{0\}$, it holds*

$$\frac{A_q^{\mathrm{opt}}(M)\|\nabla u\|_{L^2}^2 + \mathrm{Vol}_g(M)^{2/q-1}\|u\|_{L^2}^2}{\|u\|_{L^q}^2} - 1 \geq C \left(\inf_{v \in \mathcal{M}_q(A)} \frac{\|u - v\|_{W^{1,2}}}{\|u\|_{W^{1,2}}} \right)^{2+\gamma}.$$

The above stability result in Theorem 1.1 is *strong*, as we are able to control the full $W^{1,2}$-distance from a set a non empty set of extremal functions with the related Sobolev deficit. On the other hand, the constant C and the exponent γ crucially depend on the manifold M and on the value q as an outcome of the proof-strategy. A natural question at this point is whether, on a manifold (M, g), we can expect the strict binding inequality to hold and, therefore, a critical stability result. This heavily depends on the metric g, as it is known from [DD01, Corollary 2] that we always have $A_{2^*}^{\mathrm{opt}}(M, h) = \mathsf{S}_d^2$ (adding the dependence on the metric) for a suitable metric h in the conformal class of g. However, for certain manifolds this scenario is very rigid (see Proposition 5.9 and Proposition 5.12 in [Heb99]). More generally,

the continuity properties studied in [NV22, Theorem 6.2] suggests that, as soon as there is g satisfying $A_{2*}^{\mathrm{opt}}(M,g) > \mathsf{S}_d^2$, there are infinitely many others metrics.

We pass now to our second main result, which is a weaker stability results only around minimizer for $B_{2*}^{\mathrm{opt}}(M)$. Let us first comment on the non-emptyness of the set of extremal functions for $B_{2*}^{\mathrm{opt}}(M)$. When $d \geq 4$, work of Hebey [Heb99, Proposition 5.1] implies

$$(1.5) \qquad\qquad B_{2*}^{\mathrm{opt}}(M) \geq \frac{d-2}{4(d-1)} \mathsf{S}_d^2 \max_M \mathrm{R}_g,$$

where R_g denotes the scalar curvature of M. When this inequality is strict, the deep analysis of [DD01] guarantees that $\mathcal{M}_{2*}(B) \neq \varnothing$. See also Theorem 2.3 below for further details. As constant functions might not be extremals, unless $B_{2*}^{\mathrm{opt}}(M) = \beta(M)$, this result is striking. The dimensional restriction $d \geq 4$ is due to the need to consider a more general class of p-Sobolev inequalities with $p \neq 2$ when $d = 3$, see [DD01]. Under this assumption, we prove the following.

Theorem 1.2. *Let (M,g) be a closed d-dimensional Riemannian manifold, $d \geq 4$. Assume $B_{2*}^{\mathrm{opt}}(M) > \frac{d-2}{4(d-1)} \mathsf{S}_d^2 \max_M \mathrm{R}_g$. Then, there are non-negative constants C, γ, δ depending on (M,g) so that it holds*

$$\mathsf{S}_d^2 \|\nabla u\|_{L^2}^2 + B_{2*}^{\mathrm{opt}}(M)\|u\|_{L^2}^2 - 1 \geq C \left(\inf_{v \in \mathcal{M}_{2*}(B)} \frac{\|u-v\|_{W^{1,2}}}{\|u\|_{W^{1,2}}} \right)^{2+\gamma},$$

for all $u \in W^{1,2}(M)$ so that $\|u\|_{L^{2*}} = 1$ and $\|u-v\|_{W^{1,2}} < \delta\|u\|_{W^{1,2}}$ for all $v \in \mathcal{M}_{2*}(B)$ with $\|v\|_{L^{2*}} = 1$.

The above is a weaker form of stability due to the assumption forcing u to be close to the set of extremizers. Indeed, contrary to Theorem 1.1, here we cannot rely on a concentration-compactness principle (see Remark 3.3 for details). Besides, the continuous dependence $g \mapsto B_{2*}^{\mathrm{opt}}(M,g)$ studied in [BM12] implies the abundance of metrics for which (1.5) holds strict.

Given the above results, it is natural to ask if our results hold *sharp* with quadratic exponent, i.e. with $\gamma = 0$. This is the typical desirable feature in stability problems. For instance, in [CES23], respectively [ENS22], the authors show that quadratic stability might fail for the quantitative isoperimetric inequality, respectively the quantitative stability of minimizing Yamabe metrics, on specific manifolds. This degenerate phenomenon has been later analyzed by [Fra22] (see Corollary 1.4 below), and subsequently in [FP24], and [BDS23, BDS24b, BDS24a] for different functional inequalities. Our next results deal with degenerate phenomena.

Theorem 1.3. *Let $q \in (2, 2^*]$ and let (M,g) be satisfying the hypothesis of Theorem 1.1. Assume further that there are no non-constant extremal functions, i.e. $\mathcal{M}_q(A) = \{c : c \in \mathbb{R} \setminus \{0\}\}$. Then, the stability in Theorem 1.1 is degenerate, i.e. it must hold with some $\gamma > 0$.*

We point out that this result is in line with the examples studied by Frank in [Fra22], as we prove in the next corollary. The exponent $\gamma = 2$ follows from the analysis in *loc. cit.*

Corollary 1.4. *For $d > 2$, we have:*

    i) *Theorem 1.1 does not hold with $\gamma = 0$ for $q \in (2, 2^*)$ and $M = \mathbb{S}^d$;*

    ii) *Theorem 1.1 does not hold with $\gamma = 0$ for $q = 2^*$ and $M = \mathbb{S}^1\left(\frac{1}{\sqrt{d-2}}\right) \times \mathbb{S}^{d-1}$.*

Finally, in all of the above, $\gamma = 2$ is sharp.

It is unclear whether this degenerate phenomenon can happen also in Theorem 1.2, and if $M_* = \mathbb{S}^1(1/\sqrt{d-2}) \times \mathbb{S}^{d-1}$ provides again a counterexample. Moreover, we are only aware of estimates for $B_{2^*}^{\mathrm{opt}}(M_*)$ ([HV92] and [Heb99, Proposition 5.4]) and not its explicit value.

Comparison with related works. Quantitative stability results are ubiquitous in the literature. In the classical Euclidean setting, several geometric and functional inequalities have been analyzed from the viewpoint of stability with different techniques. We refer to the aforementioned references regarding the Sobolev inequality, to the surveys [Fig13, Fus15], and the references therein.

On the contrary, in the non-linear framework of Riemannian manifolds, we cannot rely on underlying symmetries of any kind as, for instance, in (1.3) for $\mathbb{S}^d$. In fact, our main argument follows a well-established approach, that is to argue via a Lyapunov-Schmidt reduction argument [Sim83] and Łojasievicz inequality [Loj65] to produce quantitative stability estimates. This fruitful interaction has been pionereed in [CES23] for the quantitative stability of the Riemannian isoperimetric inequality, and it was later employed to study stability properties of minimizing Yamabe metrics in [ENS22] (see also the recent [CK24], we refer to [Yam60] and the surveys [LP87, BM11] regarding the Yamabe problem). In the present work, we exploit the similarities between the latter and our setting.

Besides degenerate phenomena, it is natural to investigate sharp stability results with exponent $\gamma = 0$ holding *generically* in the space of Riemannian metrics. However, in contrast with [ENS22] where this follows by work Schoen [Sch89], see also [And05], the picture here seems to be more delicate. First, to formulate a generic statement, one needs to check that the strict inequalities on our optimal constants are open conditions in the set of C^3-metrics. In this direction, recall the continuity properties of $g \mapsto A_q^{\mathrm{opt}}(M,g), g \mapsto B_{2^*}^{\mathrm{opt}}(M,g)$ studied in [NV22] and [BM12], respectively. Then, the key point would be to prove that generically inside the relevant open sets of metrics, minimizers (or, more generally, non-negative critical points) of

$$\frac{A_q^{\mathrm{opt}}(M,g)\|\nabla u\|_{L^2}^2 + \mathrm{Vol}_g(M)^{2/q-1}\|u\|_{L^2}^2}{\|u\|_{L^q}^2} \quad \text{and} \quad \frac{\mathsf{S}_d^2\|\nabla u\|_{L^2}^2 + B_{2^*}^{\mathrm{opt}}(M,g)\|u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}$$

are non-degenerate. Notice again that the first always admits constant minimizers while, the second does not in general, unless $B_{2^*}^{\mathrm{opt}}(M,g) = \mathrm{Vol}_g(M)^{2/2^*-1}$ (this, however, can always happen in the conformal class of a given metric again by [DD01, Corollary 2]). Constant critical points are typically excluded from this kind of analysis, see e.g. [GM13, MP09]. Recall also that they can be pathological in this regard, as discussed in Theorem 1.3 and Corollary 1.4. Finally, even restricting our investigation to nonconstant extremal functions, it seems that standard transversality arguments [Hen05, Whi91] would require extra regularity on the dependence $g \mapsto A_q(M,g), g \mapsto B_{2^*}^{\mathrm{opt}}(M,g)$.

We conclude by mentioning that assuming Ricci curvature constraints makes it possible to investigate more sophisticated stability statements in comparison geometry. In this fashion, we recall [CMM19] for the Lévy-Gromov isoperimetric inequality, and [BBG85, Ber07, CMS23, FGS22] for the Lichnerowicz spectral gap. Concerning Sobolev inequalities, *qualitative* stability results were deduced in [NV22, NV24] under Ricci lower bounds. The main difference with this note (besides the qualitative/quantitative analysis and curvature bounds) is that in [NV22, NV24] the stability properties are studied facing lack of compactness and with explicit classes of -a priori non-extremal- functions having the radial profile of bubbles. Besides, quantitative stability results in the Hyperbolic space have been obtained for the isoperimetric problem in [BDS15], and for the sharp Poincaré-Sobolev inequality in [BGKM22a, BGKM22b].

2. Properties of Sobolev inequalities

2.1. Optimal constants and extremal functions.

In this part, we prove a key precompactness result for normalized extremizing sequences of $A_{2^*}^{\mathrm{opt}}(M)$, and mention analogous compactness properties for extremal functions of $B_{2^*}^{\mathrm{opt}}(M)$. We start with the former.

Proposition 2.1. *Let (M, g) be a compact d-dimensional Riemannian manifolds, $d > 2$. Assume that either $q < 2^*$ or*

$$q = 2^* \qquad and \qquad A_{2^*}^{\mathrm{opt}}(M) > \mathsf{S}_d^2.$$

Then, for every $(u_n) \subseteq W^{1,2}(M)$ non-constant with $\|u_n\|_{L^q} \equiv 1$ satisfying

$$\frac{\|u_n\|_{L^q} - \mathrm{Vol}_g(M)^{2/q - 1}\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \to A_q^{\mathrm{opt}}(M), \qquad as \ n \uparrow \infty,$$

it holds up to subsequence that u_n converges strongly in L^q and in $W^{1,2}$ to some $u \in \mathcal{M}_q(A)$ with $\|u\|_{L^{2^*}} = 1$. In particular, $\{u \in \mathcal{M}_q(A) \colon \|u\|_{L^q} = 1\}$ is pre-compact in the $W^{1,2}$-topology.

Proof. Up to a not relabelled subsequence, we can assume by Rellich pre-compactness that $u_n \to u$ in L^q for every $q < 2^*$. Moreover, by assumption, we have for some $\delta_n \downarrow 0$ that

$$(2.1) \qquad 1 = \|u_n\|_{L^q}^2 \geq (A_q^{\mathrm{opt}}(M) - \delta_n)\|\nabla u_n\|_{L^2}^2 + \mathrm{Vol}_g(M)^{2/q - 1}\|u_n\|_{L^2}^2,$$

In particular, u_n is $W^{1,2}$-bounded, and, up to a further subsequence, we also have $u_n \rightharpoonup u$ weakly in $W^{1,2}(M)$ to some $u \in W^{1,2}(M)$.

If $q < 2^*$, by lower-semicontinuity of the gradient term and L^2-convergence of u_n to u, from (2.1) we reach

$$1 \geq A_q^{\mathrm{opt}}(M)\|\nabla u\|_{L^2} + \mathrm{Vol}_g(M)^{2/q - 1}\|u\|_{L^2} \geq \|u\|_{L^q}^2.$$

However, $\|u\|_{L^q} = \lim_n \|u_n\|_{L^q} = 1$ so that $u \in \mathcal{M}_q(A)$, giving in turn that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$. In particular, there is convergence of $\|u_n\|_{W^{1,2}}$ to $\|u\|_{W^{1,2}}$. Being u_n already weakly converging, this is equivalent to strong $W^{1,2}$-convergence concluding the proof in this case.

If $q = 2^*$ we argue by concentration-compactness. We shall use Proposition 2.2 below and make use of the assumption in this case to rule out concentration phenomenona. If (a) in Proposition 2.2 occurs, then an analogous argument as done after (2.1) gives the conclusion. We are left to exclude (b) in Proposition 2.2 using the assumption in this case. By definition of $\alpha(M) = \mathsf{S}_d^2$ (cf. (1.4)), for every $\epsilon > 0$ there is $B_\epsilon > 0$ so that, for every $n \in \mathbb{N}$ it is possible to write

$$1 = \|u_n\|_{L^{2^*}}^2 \leq (\mathsf{S}_d^2 + \epsilon)\|\nabla u_n\|_{L^2}^2 + B_\epsilon\|u_n\|_{L^2}.$$

This and the fact that $\|u_n\|_{L^2} \to 0$ guarantee that $\liminf_n \|\nabla u_n\|_{L^2} > 0$. Rearranging now the above with (2.1), taking n to infinity, and recalling that u_n is L^2-converging to zero in this case, we get

$$A_{2^*}^{\mathrm{opt}}(M) \leq \mathsf{S}_d^2 + \epsilon.$$

We see now that for ϵ small enough we eventually reach a contradiction with $A_{2^*}^{\mathrm{opt}}(M) > \mathsf{S}_d^2$. So (b) does not occur and the proof is concluded also in this case.

Finally, the last statement follows easily from the above analysis. Indeed, a sequence $(u_n) \subseteq \{u \in \mathcal{M}_q(A) \colon \|u\|_{L^q} = 1\}$ is either eventually constant, or not. In the first situation, the renormalization guarantees that, up to a non-relabelled subsequence, $u_n \equiv cst.$ and we are done. The second situation is instead a particular case of what was addressed before. The proof is therefore concluded. $\square$

In what follows we prove a dichotomy result based on concentration compactness arguments that were needed in the above proof. This was studied in [NV22, Theorem 6.1] in the spirit of [Lio84, Lio85] but in a different setting (converging metric measure spaces with synthetic Ricci bounds). We include here a quick proof with the core of the argument to be self-contained.

Proposition 2.2. *Let (M, g) be a compact and complete d-dimensional Riemannian manifold, $d > 2$. Suppose that it holds*

$$(2.2) \qquad \|u\|_{L^{2*}}^2 \leq A\|\nabla u\|_{L^2}^2 + B\|u\|_{L^2}^2, \qquad \forall u \in W^{1,2}(M),$$

for some constants $A, B > 0$, and there exist a sequence $(u_n) \subset W^{1,2}(M) \setminus \{0\}$ satisfying

$$\|u_n\|_{L^{2*}}^2 \geq A_n\|\nabla u_n\|_{L^2}^2 + B_n\|u_n\|_{L^2}^2,$$

for some sequences $A_n \to A$, $B_n \to B$. Then, setting $\tilde{u}_n := u_n\|u_n\|_{L^{2*}}^{-1}$, there exists a non-relabeled subsequence such that only one of the following holds:

(a) *$\tilde{u}_n$ converges L^{2*}-strong to a function $u \in W^{1,2}(M)$;*
(b) *$\|\tilde{u}_n\|_{L^2} \to 0$ and there exists $p \in M$ so that $|\tilde{u}_n|^{2*}\mathrm{Vol}_g \rightharpoonup \delta_p$ in duality with $C(M)$.*

Proof. By a scaling argument, we can assume $\|u_n\|_{L^{2*}} \equiv 1$. As $(A_n, B_n) \to (A, B)$, we directly deduce that u_n is $W^{1,2}$-bounded. Hence, up to a not relabeled subsequence, we have that u_n converges to some $u \in W^{1,2}(M)$ weakly in $W^{1,2}$ and in L^{2*}, and strongly in L^2, to some function $u \in W^{1,2}(M)$ and, by tightness as M is compact, that $|u_n|^{2*}\mathrm{Vol}_g \rightharpoonup \nu$, as well as $|\nabla u_n|^2\mathrm{Vol}_g \rightharpoonup \mu$, in duality with $C(M)$ for a probability measure μ and for a finite non-negative measure ν. We have [Lio84, Lio85] (see also the arguments [NV22, Lemma 6.6]) that there is a countable set of indices J, points $(x_j)_{j \in J} \subseteq M$ and weights $(\mu_j)_{j \in J}, (\nu_j)_{j \in J} \subseteq \mathbb{R}^+$, satisfying

$$\nu = |u|^{2*}\mathrm{Vol}_g + \sum_{j \in J} \nu_j \delta_{x_j},$$

$$\mu \geq |\nabla u|^2\mathrm{Vol}_g + \sum_{j \in J} \mu_j \delta_{x_j},$$

$$\nu_j^{2/2*} \leq A\mu_j, \qquad \sum_{j \in J} \nu_j^{2/2*} < \infty.$$

We can therefore estimate

$$
\begin{aligned}
1 = \lim_{n \uparrow \infty} \|u_n\|_{L^{2*}}^2 &\geq \liminf_{n \uparrow \infty} A_n\|\nabla u_n\|_{L^2}^2 + B_n\|u_n\|_{L^2}^2 \\
&\geq A\mu(M) + B\|u\|_{L^2}^2 \\
&\geq A\|\nabla u\|_{L^2}^2 + \sum_{j \in J} A\mu_j + B\|u\|_{L^2}^2 \\
&\geq \left(\int |u|^{2*} d\mathrm{Vol}_g\right)^{2/2*} + \sum_{j \in J} \nu_j^{2/2*} \\
&\geq \left(\int |u|^{2*} d\mathrm{Vol}_g + \sum_{j \in J} \nu_j\right)^{2/2*} = \nu(M)^{2/2*} = 1,
\end{aligned}
$$

having used that u satisfies (2.2) with A, B, the properties of μ, ν, as well as the fact that $t \mapsto t^{2/2*}$ is strictly concave. However, since all inequalities must be equalities, we have that either $\|u\|_{L^{2*}} = 1$ and all $\nu_j = 0$ for every $j \in J$, or $u = 0$ and there is $j_0 \in J$ so that $\nu_{j_0} = 1$

and $\nu_j = 0$ for all $j \neq j_0$. The first case is precisely conclusion (a), as weak convergence in L^{2^*} together with convergence of the L^{2^*} norms implies strong convergence. If instead $u = 0$, then (b) occurred implying at the same time that u_n is L^2-converging to zero, and $|u_n|^{2^*} \mathrm{Vol}_g$ is weakly converging to a Dirac mass centred at $p = x_{j_0}$. Having analysed all the possibilities, the proof is now concluded. □

We now discuss the case of $B_{2^*}^{\mathrm{opt}}(M)$. Recall that thanks to [Heb99, Proposition 5.1], we have the lower bound (1.5). Notice that, even arguing with Proposition 2.2 for minimizing sequences of $B_{2^*}^{\mathrm{opt}}(M)$, we cannot exclude the outcome (b), as this time $A = \mathsf{S}_d^2$ and the convergence to zero of the L^2-norm makes it impossible to exploit (1.5). However, sufficient criteria guaranteeing that $\mathcal{M}_{2^*}(B)$ is not empty have been analysed in [DD01, Theorem 1 and Corollary 1].

Theorem 2.3. *Let (M, g) be a d-dimensional compact Riemannian manifold with $d \geq 4$. Then, at least one of the following is true*

  i) $\mathcal{M}_{2^*}(B) \neq \varnothing$;
  ii) $B_{2^*}^{\mathrm{opt}}(M) = \frac{d-2}{4(d-1)} \mathsf{S}_d^2 \max_M \mathrm{R}_g$.

In particular, if strict inequality holds in (1.5), *then $\mathcal{M}_{2^*}(B) \neq \varnothing$.*

In lower dimensions, a related result was studied in [DD01] by considering p-Sobolev inequalities with $p \neq 2$. As we stick to $p = 2$ in this note, we do not pursue this any further. The above is a dichotomy result achieved via a delicate study of blow-up phenomena of concentrating sequences of solutions of partial differential equations. In particular, strict inequality in (1.5) guarantees compactness. More generally, we recall that $\{u \in \mathcal{M}_{2^*}(B) : \|u\|_{L^{2^*}} = 1\}$ is pre-compact in L^{2^*}-topology if strict inequality holds in (1.5). This result is commonly attributed to the analysis contained in [DD01], see the introduction in [BM12], as well as [BM12, Corollary 1.4] for generalizations that, as a by-product, imply this fact. Therefore, arguing exactly as done in Proposition 2.1, we deduce

(2.3) $\{u \in \mathcal{M}_{2^*}(B) : \|u\|_{L^{2^*}} = 1\}$ is $W^{1,2}$ pre-compact, if $d \geq 4$ and (1.5) holds strict.

Let us discuss a relevant class of manifolds when the above discussion applies. Notice that i) must occur in Theorem 2.3 if either $\mathrm{R}_g \leq 0$, or R_g is constant. The first case being clear, we discuss the second one. The case where (M, g) is conformally equivalent to the round sphere has been fully investigated in [Heb99, Theorem 5.7]. In the remaining case, the resolution of the Yamabe problem [Yam60] by combination of [Tru68, Aub76a, Sch84] guarantees that

$$Y(M) < \mathsf{S}_d^{-2},$$

having denoted $Y(M)$ the Yamabe constant of (M, g) that is attained by some $u \in W^{1,2}(M)$

$$\|u\|_{L^{2^*}}^2 Y(M) = \|\nabla u\|_{L^2}^2 + \frac{d-2}{4(d-1)} \mathrm{R}_g \|u\|_{L^2},$$

having used that R_g is constant. Thus, there holds

$$\|u\|_{L^2}^2 \frac{B_{2^*}^{\mathrm{opt}}(M)}{\mathsf{S}_d^2} + \|\nabla u\|_{L^2}^2 \geq \|u\|_{L^{2^*}} \mathsf{S}_d^{-2} > \|u\|_{L^{2^*}} Y(M) = \|\nabla u\|_{L^2}^2 + \frac{d-2}{4(d-1)} \mathrm{R}_g \|u\|_{L^2}.$$

Rearranging, we get that ii) cannot occur.

2.2. **First and second-order variations.** In this part, we study functional properties of abstract Sobolev quotients. This investigation is analogous to [ENS22, Section 2]. We start setting up some notation. We fix $q \in (2, 2^*]$. We omit the dependence on it of various quantities to lighten the notation. For $\delta > 0$, set

$$\mathcal{B} := \{u \in W^{1,2}(M) \colon u \geq 0, \|u\|_{L^q} = 1\}, \qquad \mathcal{B}_\delta(v) := \{u \in \mathcal{B} \colon \|u - v\|_{W^{1,2}} \leq \delta\}.$$

Arguing as in [ENS22, Lemma 2.1], $\mathcal{B} \subseteq W^{1,2}$ is a Banach submanifold and the tangent space at $u \in \mathcal{B}$ is given by

$$T_u\mathcal{B} := \left\{v \in W^{1,2}(M) \colon \int u^{q-1} v \, d\mathrm{Vol}_g = 0\right\}.$$

Since $W^{1,2}(M)$ is a Hilbert space, we denote by $\pi_{T_u\mathcal{B}}$ the L^2-projection operator defined by

$$\pi_{T_u\mathcal{B}}(\varphi) := \varphi - \left(\int u^{q-1} \varphi \, d\mathrm{Vol}_g\right) u.$$

We compute now the first and second variations of the functional arising by the Sobolev inequality (cf. [ENS22, Lemma 2.1]).

Lemma 2.4. *Let (M, g) be a compact d-dimensional Riemannian manifold for $d > 2$. Set for $u \in W^{1,2}(M) \setminus \{0\}$ and $A, B > 0, q \in (2, 2^*]$:*

$$\mathcal{Q}_q(u) := \frac{A\|\nabla u\|_{L^2}^2 + B\|u\|_{L^2}^2}{\|u\|_{L^q}^2}.$$

For $u \in \mathcal{B}$, the first and second variation of $\mathcal{Q}_q$ at u restricted to $T_u\mathcal{B}$ are denoted by $\nabla_\mathcal{B}\mathcal{Q}_q(u), \nabla_\mathcal{B}^2\mathcal{Q}_q(u)$ and are respectively given by

$$\nabla_\mathcal{B}\mathcal{Q}_q(u)[\varphi] = 2\int A\nabla u\nabla\varphi + Bu\varphi \, d\mathrm{Vol}_g,$$

and

$$\nabla_\mathcal{B}^2\mathcal{Q}_q(u)[\varphi, \eta] = 2\int A\nabla\varphi\nabla\eta + B\varphi\eta \, d\mathrm{Vol}_g - 2(q-1)\mathcal{Q}_q(u)\int u^{q-2}\varphi\eta \, d\mathrm{Vol}_g,$$

for all $\varphi, \eta \in W^{1,2}(M)$ (omitting the projections $\pi_{T_u\mathcal{B}}$ inside the integrals).

Moreover, the following properties hold:

  i) *$w \mapsto \dfrac{\nabla_\mathcal{B}^2\mathcal{Q}_q(w)[\varphi, \cdot]}{\|\varphi\|_{C^{2,\alpha}}}$ is continuous from $C^{2,\alpha} \cap \mathcal{B} \to C^{0,\alpha}$ with uniform modulus of continuity over $\varphi \in C^{2,\alpha}$;*

  ii) *$w \mapsto \dfrac{\nabla_\mathcal{B}^2\mathcal{Q}_q(w)[\varphi, \eta]}{\|\varphi\|_{W^{1,2}}\|\eta\|_{W^{1,2}}}$ is continuous from $\mathcal{B} \to \mathbb{R}$ with uniform modulus of continuity over $\varphi, \eta \in W^{1,2}$.*

Proof. We fix any $q \in (2, 2^*]$ and $u \in \mathcal{B}$, and consider any $\varphi \in C^\infty(M), \epsilon \in (-1, 1)$. Set

$$u_\epsilon := (u + \epsilon\varphi)/I_\epsilon, \qquad I_\epsilon := \|u + \epsilon\varphi\|_{L^q}.$$

We claim that u_ϵ is eventually for $|\epsilon|$ small, well defined. Indeed, from the estimate $\big||x + \epsilon y|^q - |x|^q\big| \leq q|\epsilon y|\big||x + \epsilon y|^{q-1} + |x|^{q-1}\big|$ together with the fact that $u, \varphi \in L^q$, we reach that $I_\epsilon \to 1$ as $\epsilon \to 0$. For future use, we notice

$$(2.4) \quad \lim_{\epsilon \to 0} \frac{1 - I_\epsilon^2}{\epsilon} = \frac{2}{q}\left(\int |u|^q \, d\mathrm{Vol}_g\right)^{\frac{2}{q}-1} \lim \frac{\int |u|^q - |u + \epsilon\varphi|^q \, d\mathrm{Vol}_g}{\epsilon} = -2\int u^{q-1}\varphi \, d\mathrm{Vol}_g,$$

having used dominated convergence theorem combined with the same estimate as before for the last equality (recall also that $\|u\|_{L^q} = 1$). We start therefore computing the first variation

$$\nabla \mathcal{Q}_q(u)[\varphi] := \lim_{\epsilon \to 0} \frac{\mathcal{Q}_q(u_\epsilon) - \mathcal{Q}_q(u)}{\epsilon}.$$

By polarization, we derive

$$\frac{\mathcal{Q}_q(u_\epsilon) - \mathcal{Q}_q(u)}{\epsilon} = \frac{\int A|\nabla(u + \epsilon\varphi)|^2 + B|u + \epsilon\varphi|^2 \, d\mathrm{Vol}_g}{\epsilon I_\epsilon^2} - \frac{\int A|\nabla u|^2 + B|u|^2 \, d\mathrm{Vol}_g}{\epsilon}$$

$$= \frac{1}{\epsilon}\Big[\frac{1}{I_\epsilon^2} - 1\Big]\mathcal{Q}_q(u) + \frac{2}{I_\epsilon^2}\int A\nabla u \nabla \varphi + Bu\varphi \, d\mathrm{Vol}_g$$

$$+ \frac{\epsilon}{I_\epsilon^2}\int A|\nabla\varphi|^2 + B\varphi^2 \, d\mathrm{Vol}_g.$$

Hence, recalling that $I_\epsilon \to 1$ (2.4), we deduce

$$\nabla \mathcal{Q}_q(u)[\varphi] = 2\int A\nabla u \cdot \nabla\varphi + Bu\varphi \, d\mathrm{Vol}_g - 2\mathcal{Q}_q(u)\int u^{q-1}\varphi \, d\mathrm{Vol}_g.$$

So, the restriction to $T_u\mathcal{B}$ satisfies

$$\nabla_\mathcal{B} \mathcal{Q}_q(u)[\varphi] = 2\int A\nabla u \cdot \nabla\pi_{T_u\mathcal{B}}(\varphi) + Bu\pi_{T_u\mathcal{B}}(\varphi) \, d\mathrm{Vol}_g.$$

We compute now the second variation for

$$\nabla^2 \mathcal{Q}_q(u)[\varphi, \varphi] := \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0}\mathcal{Q}_q(u_\epsilon).$$

Starting from the computations already performed for the first variation, we have

$$\frac{\mathcal{Q}_q(u_\epsilon) - 2\mathcal{Q}_q(u) + \mathcal{Q}_q(v_{-\epsilon})}{\epsilon^2} = \frac{1}{\epsilon^2}\Big(\frac{1}{I_\epsilon^2} - 2 + \frac{1}{I_{-\epsilon}^2}\Big)\mathcal{Q}_q(u)$$

$$+ \frac{2}{\epsilon}\Big(\frac{1}{I_\epsilon^2} - \frac{1}{I_{-\epsilon}^2}\Big)\int A\nabla u \nabla\varphi + Bu\varphi \, d\mathrm{Vol}_g$$

$$+ \Big(\frac{1}{I_\epsilon^2} + \frac{1}{I_{-\epsilon}^2}\Big)\int A|\nabla\varphi|^2 + B\varphi^2 \, d\mathrm{Vol}_g.$$

Hence, taking the limit, we reach

$$\nabla^2 \mathcal{Q}_q(u)[\varphi, \varphi] = -2(q-1)\mathcal{Q}_q(u)\int u^{q-2}\varphi^2 \, d\mathrm{Vol}_g$$

$$+ 8\Big(\int u^{q-1}\varphi \, d\mathrm{Vol}_g\Big)\int A\nabla u \cdot \nabla\varphi + Bu\varphi \, d\mathrm{Vol}_g$$

$$+ 2\int A|\nabla\varphi|^2 + B\varphi^2 \, d\mathrm{Vol}_g.$$

So, the restriction to $T_u\mathcal{B}$ reads

$$\nabla_\mathcal{B}^2 \mathcal{Q}_q(u)[\varphi, \varphi] = 2\int A|\nabla\pi_{T_u\mathcal{B}}(\varphi)|^2 + B\pi_{T_u\mathcal{B}}(\varphi)^2 \, d\mathrm{Vol}_g$$

$$- 2(q-1)\mathcal{Q}_q(u)\int u^{q-2}\pi_{T_u\mathcal{B}}(\varphi)^2 \, d\mathrm{Vol}_g.$$

Recalling that $\mathcal{Q}_q(1) = cst.$, the desired second variation formula hold for all couples $\varphi, \eta \in W^{1,2}(M)$ by polarization.

We now verify i). For every $w \in C^{2,\alpha}$, we define $L_w(\varphi) := -2A\Delta\varphi + B\varphi - 2(q-1)\mathcal{Q}_q(w)w^{q-2}\varphi$. Then, for every $w, v \in C^{2,\alpha}$ we have for some $C > 0$

$$\|L_w(\varphi) - L_v(\varphi)\|_{C^{0,\alpha}} \le C\big(\|w^{q-2}\varphi\|_{C^{0,\alpha}}|\mathcal{Q}_q(w) - \mathcal{Q}_q(v)| + \||\varphi|w^{q-2} - v^{q-2}|\|_{C^{0,\alpha}}\big).$$

Hence, taking into account the continuity of $t \mapsto t^{q-2}$ and that of $w \mapsto \mathcal{Q}_q(w)$ in $C^{2,\alpha}$, it holds

$$\|L_w(\varphi) - L_v(\varphi)\|_{C^{0,\alpha}} \le f(\|w-v\|_{C^{2,\alpha}})\|\varphi\|_{C^{2,\alpha}},$$

for some modulus of continuity $t \mapsto f(t)$ with $f(t) \downarrow 0$ as $t \downarrow 0$ uniform on φ. To prove i), we thus need to prove continuity properties of $w \mapsto \pi_{T_w\mathcal{B}}$ as a map from $C^{2,\alpha} \cap \mathcal{B} \to \mathcal{L}(C^{2,\alpha}, C^{2,\alpha})$. This is implied by the estimate

$$\left|\pi_{T_w\mathcal{B}}(\varphi) - \pi_{T_v\mathcal{B}}(\varphi)\right| = \left|w - v + \int (v^{q-1} - w^{q-1})\varphi\, d\mathrm{Vol}_g\right| \le C\|w - w\|_{C^{0,\alpha}}\|\varphi\|_{C^{2,\alpha}}.$$

We now prove ii) and conclude. For every $\varphi \in W^{1,2}(M)$, and interpreting by slight abuse of notation $L_w(\varphi)$ in distributional sense, we estimate similarly as before

$$\big\|L_w(\varphi) - L_v(\varphi)\big\|_{H^{-1}} \le C\big(\|w^{q-2}\varphi\|_{H^{-1}}|\mathcal{Q}_q(w) - \mathcal{Q}_q(v)| + \||\varphi|w^{q-2} - v^{q-2}|\|_{H^{-1}}\big).$$

Then, by Hölder and the Sobolev inequality, we get

$$\|w^{q-2}\varphi\|_{H^{-1}} \le \|w^{q-2}\|_{L^{q/(q-2)}}\|\varphi\|_{L^q} \le C\|w\|_{L^q}^{q-2}\|\varphi\|_{W^{1,2}},$$

for some $C > 0$ and, using $|a - b|^{\frac{1}{q-2}} \le c_q|a^{\frac{1}{q-2}} - b^{\frac{1}{q-2}}|$, we estimate similarly

$$\||\varphi|w^{q-2} - v^{q-2}|\|_{H^{-1}} \le c_q\|w - v\|_{L^q}^{q-2}\|\varphi\|_{L^q} \le f\big(\|w - v\|_{W^{1,2}}\big)\|\varphi\|_{W^{1,2}},$$

for some modulus of continuity $t \mapsto f(t)$ with $f(t) \downarrow 0$ as $t \downarrow 0$ uniform on φ. Recall also that $\mathcal{Q}_q(\cdot)$ is continuous in $W^{1,2}$. From here, the conclusion follows by continuity properties of $w \mapsto \pi_{T_w\mathcal{B}}$ as a mapping $\mathcal{B} \to \mathcal{L}(W^{1,2}, W^{1,2})$. $\qquad\square$

We also have the crucial property that $\mathcal{Q}_q(\cdot)$ is analytic in the sense of [CCR15, Lemma 6] whose proof carries over in our setting.

Proposition 2.5. *Under the same assumptions of Lemma 2.4, the map $u \mapsto \mathcal{Q}_q(u)$ is analytic on $\{u \in C^{2,\alpha}(M), u > 0\}$, in the sense that for every $u_0 \in C^{2,\alpha}(M)$ with $u_0 > 0$, there is $\epsilon > 0$, and bounded multilinear operators $\mathcal{Q}_q^{(k)}\colon C^{2,\alpha}(M)^{\times k} \to \mathbb{R}$ for every $k \ge 0$, such that if $\|u - u_0\|_{C^{2,\alpha}(M)} < \epsilon$, then*

$$\sum_{k=0}^{\infty} \|\mathcal{Q}_q^{(k)}\| \cdot \|u - u_0\|_{C^{2,\alpha}(M)}^k < \infty,$$

as well as

$$\mathcal{Q}_q(u) = \sum_{k=0}^{\infty} \mathcal{Q}_q^{(k)}(u - u_0, u - u_0, \ldots, u - u_0), \qquad in\ C^{2,\alpha}(M).$$

2.3. **Lyapunov-Schmidt reduction.** The aim of this section is to discuss the Lyapunov-Schmidt reduction associated with a general Sobolev quotient that, intuitively, reduces the infinite dimensional problem to a more manageable finite dimensional one.

Suppose that

$$(2.5) \qquad \inf_{u \in W^{1,2}(M) \setminus \{0\}} \mathcal{Q}_q(u) = \inf_{u \in W^{1,2}(M) \setminus \{0\}} \frac{A\|\nabla u\|_{L^2}^2 + B\|u\|_{L^2}^2}{\|u\|_{L^q}^2} = 1.$$

for $A, B > 0$ and $q \in (2, 2^*]$ either as in Theorem 1.1 or Theorem 1.2. We shall not specify their value to unify the discussion here. Consider the set of normalized extremal functions for (2.5)

$$\mathcal{M}^1 := \{u \in \mathcal{B} \colon \mathcal{Q}_q(u) = 1\},$$

and the set of normalized critical points $\mathcal{CP}^1 \subseteq W^{1,2}(M)$ as the collection of $u \in \mathcal{B}$ so that

$$\int_M A\nabla u \cdot \nabla \varphi + Bu\varphi \, d\mathrm{Vol}_g = \mathcal{Q}_q(u) \int_M u^{q-1}\varphi \, d\mathrm{Vol}_g, \qquad \forall \varphi \in C^\infty(M).$$

The sets $\mathcal{M}^1$ and $\mathcal{CP}^1$ depend on q, A, B and coincide with those discussed in the Introduction. Later, in the main proofs, these constants will be chosen. Furthermore, we can regard $\nabla_{\mathcal{B}}^2 \mathcal{Q}_q(u)[\cdot, \cdot]$ as a mapping $T_u\mathcal{B} \subseteq W^{1,2}(M) \to H^{-1}(M)$. Since the operator associated with this mapping is elliptic on a compact manifold, we know that

$$K := \mathrm{Ker}\, \nabla_{\mathcal{B}}^2 \mathcal{Q}_q(u),$$

has finite dimension, say $\dim(K) = l < \infty$. We then denote by $K^\perp$ its L^2-orthogonal complement in $W^{1,2}(M)$. Given the properties of $\mathcal{Q}_q(\cdot)$, a proof of the following result follows by [Sim96, Section 3.12], as well as [ENS22, Appendix A] replacing the use of [ENS22, Lemma 2.1] with Lemma 2.4. Recall also that any $v \in \mathcal{M}^1$ is $C^{2,\alpha}$-regular and it s either strictly positive or negative, by elliptic regularity (see, e.g., [Dru00]). We also refer the reader to the instructive finite dimensional version for smooth functions in [Sim96, Appendix 3.16.1].

Proposition 2.6 (Lyapunov-Schmidt reduction). *Let (M, g) be a d-dimensional compact Riemannian manifold, $d > 2$ with a C^3 metric. Suppose that (2.5) holds with $A, B > 0$ and $q \in (2, 2^*]$. Fix $v \in \mathcal{M}^1$. There are an open neighborhood U of 0, such that $U \subset K$, and a map $F \colon U \to K^\perp$ satisfying $F(0) = 0$, and $\nabla F(0) = 0$ such that the following hold.*

i) *Let $\mathfrak{q} \colon U \to \mathbb{R}$ be the analytic function defined by $\mathfrak{q}(\varphi) = \mathcal{Q}_q(v + \varphi + F(\varphi))$. Then*

$$\mathcal{L} := \{v + \varphi + F(\varphi); \ \varphi \in U\} \subset \mathcal{B},$$

and

$$\nabla_{\mathcal{B}} \mathcal{Q}_q(v + \varphi + F(\varphi)) = \pi_K \nabla_{\mathcal{B}} \mathcal{Q}_q(v + \varphi + F(\varphi)) = \nabla \mathfrak{q}(\varphi).$$

Furthermore, $\varphi \mapsto \mathfrak{q}(\varphi)$ is real analytic;

ii) *There exists $\delta > 0$, depending on v such that for any $u \in \mathcal{B}_\delta(v)$, we have $\pi_K(u - v) \in U$. Furthermore, if $u \in \mathcal{CP}^1 \cap \mathcal{B}_\delta(v)$, then*

$$u = v + \pi_K(u - v) + F(\pi_K(u - v));$$

iii) *There exists $C > 0$, such that for all $\varphi \in U$ and $\eta \in K$, we have the bound*

$$\|\nabla F(\varphi)[\eta]\|_{C^{2,\alpha}} \le C\|\eta\|_{C^{0,\alpha}}.$$

3. Quantitative stability

Aim of this section is to prove first a local quantitative stability, and then pass to the proof of our main quantitative stability results.

3.1. Local quantitative stability.

We shall continue assuming here the validity of (2.5), without specifying the value of $A, B > 0$ and $q \in (2, 2^*]$. Recall now K, F from Proposition 2.6 and the definitions of $\mathcal{M}^1$ and $\mathcal{CP}^1$ given above.

Associated to the Lyapunov-Schmidt reduction, we have the following notions ([AS88] see also [CCR15]). For $\delta > 0, v \in \mathcal{M}^1$ given by Proposition 2.6 and $u \in \mathcal{B}_\delta(v)$, we define the *Lyapunov-Schmidt projection operator*

$$u_\mathcal{L} := v + \pi_K(u - v) + F(\pi_K(u - v)).$$

We can thus distinguish between the following notions:

- We say that v is *non-degenerate* if $\mathrm{Ker}\nabla^2_\mathcal{B}\mathcal{Q}_q(v) = \{0\}$. Hence, $u_\mathcal{L} = v$ in this case for every $u \in \mathcal{B}_\delta(v)$;
- We say that $v \in \mathcal{CP}^1$ is *integrable*, provided that for every $\varphi \in \mathrm{Ker}\nabla^2_\mathcal{B}\mathcal{Q}_q(v)$, there is a one-parameter family of functions $(v_t)_{t\in(-\delta,\delta)}$ with $v_0 = v$, and $\frac{dv_t}{dt}|_{t=0} = \varphi$, as well as $v_t \in \mathcal{CP}^1$ for all t sufficiently small. Conversely, we say that v is non *non-integrable* if this property does not hold.

Arguing exactly as in [ENS22, Lemma 2.4] in the present setting, we have that if v is integrable, then $\mathfrak{q}$ as given in Proposition 2.6 is constant in a neighborhood of $\{0\} \in K$, and when v is an integrable minimizer also

$$(3.1) \qquad\qquad \mathcal{M}^1 \cap \mathcal{B}_\delta(v) = \mathcal{L},$$

where $\mathcal{L}$ is the set given in Proposition 2.6. We define, for $v \in \mathcal{M}^1$ and $\delta > 0$, the local distance

$$d_\delta(u, \mathcal{M}^1) := \inf\left\{\frac{\|u - \tilde{v}\|_{W^{1,2}}}{\|u\|_{W^{1,2}}} : \tilde{v} \in \mathcal{M}^1 \cap \mathcal{B}_\delta(v)\right\}.$$

Recall also the following inequality due to [Loj65].

Proposition 3.1 (Łojasievicz inequality). *Let* $\mathfrak{q}: \mathbb{R}^l \to \mathbb{R}$ *be a real analytic map and assume* $\nabla\mathfrak{q}(0) = 0$. *Then, there exist* $\delta > 0$, $C > 0$, *and* $\gamma > 0$ *depending on* $\mathfrak{q}$ *and the critical point* 0, *such that for all* $z \in B(0, \delta)$, *it holds*

$$|\mathfrak{q}(z) - \mathfrak{q}(0)| \geq C \left(\inf\{|\varphi - \overline{\varphi}|; \; \overline{\varphi} \in B_\delta(0), \nabla\mathfrak{q}(\overline{\varphi}) = 0\}\right)^{2+\gamma}.$$

We can prove now the main local stability estimate.

Proposition 3.2 (Local quantitative stability). *Let* (M, g) *be a closed* d-*dimensional Riemannian manifold,* $d > 2$. *Suppose that* (2.5) *holds with* $A, B > 0$ *and* $q \in (2, 2^*]$. *Fix* $v \in \mathcal{M}^1$. *Then, there exist positive constants* C, γ, δ *depending on* v *such that*

$$\mathcal{Q}_q(u) - 1 \geq Cd_\delta(u, \mathcal{M}^1)^{2+\gamma}, \qquad \forall u \in \mathcal{B}_\delta(v).$$

Moreover, if v *is integrable or non-degenerate, we may take* $\gamma = 0$.

Proof. In this proof, C is a general positive constant depending on v (in particular on (M, g) and the value of q) whose value might change from line to line.

Taking $\delta > 0$ in Proposition 2.6 smaller if necessary, we can ensure that for any $\epsilon > 0$, we have $\|u_\mathcal{L} - u\|_{W^{1,2}} \le \epsilon$, as well as $\|u_\mathcal{L} - v\|_{W^{1,2}} \le \epsilon$. In particular, we can write

$$\mathcal{Q}_q(u) - 1 = (\mathcal{Q}_q(u) - \mathcal{Q}_q(u_\mathcal{L})) + (\mathcal{Q}_q(u_\mathcal{L}) - 1) =: \mathrm{I} + \mathrm{II}.$$

We analyse the two term separately. For the first term, write $u = u_\mathcal{L} + u^\perp$, and Taylor expand the functional to get

$$(3.2) \qquad \begin{aligned} \mathrm{I} &= \nabla_\mathcal{B} \mathcal{Q}_q(u_\mathcal{L})[u^\perp] + \frac{1}{2}\nabla_\mathcal{B}^2 \mathcal{Q}_q(\zeta)[u^\perp, u^\perp] \\ &= \frac{1}{2}\nabla_\mathcal{B}^2 \mathcal{Q}_q(v)[u^\perp, u^\perp] + o(1)\|u^\perp\|_{W^{1,2}}^2 \ge C\lambda_1\|u^\perp\|_{W^{1,2}}^2 = C\|u - u_\mathcal{L}\|_{W^{1,2}}^2, \end{aligned}$$

where λ_1 is the smallest non-zero eigenvalue of the Hessian of $\mathcal{Q}_q$. Here ζ is a geodesic point in $\mathcal{B}$ between u and $u_\mathcal{L}$ and $o(1)$ is a quantity that goes to zero as $\|u - v\|_{W^{1,2}}$ goes to zero. For the second equality, we used the fact that $\nabla_\mathcal{B}\mathcal{Q}_q(u_\mathcal{L})[u^\perp] = 0$, as well as continuity of the Hessian, cf. Lemma 2.4 property (ii). The last inequality holds by taking $\delta > 0$ small enough.

We can now turn to the second term II specializing the discussion depending on v being non-degenerate, integrable or non-integrable. First, if v is non-degenerate, we have $u_\mathcal{L} = v$, so $\mathrm{II} = 0$ and (3.2) immediatly yields the conclusion. When v is integrable, we infer $\mathcal{Q}_q(u_\mathcal{L}) = \mathfrak{q}(0) = Q_q(v)$, where the first equality follows from $\mathfrak{q}$ being constant in a neighborhood of the origin (cf. above (3.1)). In particular, also in this situation $\mathrm{II} = 0$ and since $u_\mathcal{L} \in \mathcal{M}^1 \cap B_\delta(v)$ the estimate (3.2) combined with the simple observation

$$\|u - u_\mathcal{L}\|_{W^{1,2}}^2 \ge \left(\inf\left\{\|u - \tilde{v}\|_{W^{1,2}} : \tilde{v} \in \mathcal{M}^1 \cap B_\delta(v)\right\}\right)^2,$$

immediately give the conclusion. It remains to handle the case when v is non-integrable. Denoting $\varphi = \pi_K(u - v)$, we deduce

$$\begin{aligned} \mathrm{II} = \mathfrak{q}(\varphi) - \mathfrak{q}(0) &\ge C\left(\inf\{|\varphi - \overline{\varphi}|;\ \overline{\varphi} \in K \cap B_\delta(0), \nabla\mathfrak{q}(\overline{\varphi}) = 0\}\right)^{2+\gamma} \\ &\ge C\inf\{\|u_\mathcal{L} - \tilde{v}\|_{W^{1,2}};\ \tilde{v} \in \mathcal{M}^1 \cap \mathcal{B}_\delta(v)\}^{2+\gamma}, \end{aligned}$$

thanks to Proposition 3.1, and where we used the second and third properties of Proposition 2.6, while for the last inequality we argued as in [ENS22, page 405]. Thus, the conclusion of the proof follows by noticing

$$\begin{aligned} \mathcal{Q}_q(u) - 1 &\ge C\|u - u_\mathcal{L}\|_{W^{1,2}}^2 + C\inf\{\|u_\mathcal{L} - \hat{v}\|_{W^{1,2}};\ \hat{v} \in \mathcal{M}^1 \cap \mathcal{B}_\delta(v)\}^{2+\gamma} \\ &\ge C\left(\|u - u_\mathcal{L}\|_{W^{1,2}} + \inf\{\|u_\mathcal{L} - \hat{v}\|_{W^{1,2}};\ \hat{v} \in \mathcal{M}^1 \cap \mathcal{B}_\delta(v)\}\right)^{2+\gamma} \\ &\ge C\left(\inf\left\{\|u - \tilde{v}\|_{W^{1,2}} : \tilde{v} \in \mathcal{M}^1 \cap B_\delta(v)\right\}\right)^{2+\gamma}. \end{aligned}$$

$\square$

3.2. Proof of Theorem 1.1 and Theorem 1.2.

We start with the proof of Theorem 1.1 and then explain how to modify it to obtain Theorem 1.2.

Fix any $q \in (2, 2^*]$ and set here $A = A_q^{\mathrm{opt}}(M)$ and $B = \mathrm{Vol}_g(M)^{2/q-1}$. We claim that it is enough to prove Theorem 1.1 for every $u \in \mathcal{B} = \{u \in W^{1,2}(M) : \|u\|_{L^q} = 1\}$. Indeed, the main estimate is zero-homogeneous in u and

$$\inf\{\|u - v\|_{W^{1,2}} : v \in \mathcal{M}_q(A) \cap \mathcal{B}\} \ge \inf\left\{\frac{\|u - v\|_{W^{1,2}}}{\|u\|_{W^{1,2}}} : v \in \mathcal{M}_q(A)\right\} =: d(u, \mathcal{M}_q(A)).$$

Thus, fix $v \in \mathcal{M}_q(A)$ with $\|v\|_{L^q(M)} = 1$. Let us invoke Proposition 3.2 with $\mathcal{M}^1 = \mathcal{M}_q(A) \cap \mathcal{B}$ to get the existence of constants $\gamma(v), \delta(v)$, and $C(v)$. In particular, by compactness of

$\mathcal{M}_q(A) \cap \mathcal{B}$, c.f. Proposition 2.1, we can take a finite subcover (indexed by I) of the open cover of balls $\mathcal{B}(v, \delta(v)/2)$. Notice that, in the case $q = 2^*$, our extra assumption on $A_{2^*}^{\mathrm{opt}}(M)$ plays a crucial role. Set now

$$\delta_0 = \min_I \delta(v_i)/2, \quad \gamma_0 = \max_I \gamma(v_i)/2, \quad \text{and} \quad C_0 = \min_I C(v_i).$$

We now consider two cases: let $u \in \mathcal{B}$ satisfy either $d(u, \mathcal{M}_q(A) \cap \mathcal{B}) \leq \delta_0/4$, or $d(u, \mathcal{M}_q(A) \cap \mathcal{B}) > \delta_0/4$. Assuming the former, there exists $i \in I$ such that $\|v_i - u\|_{W^{1,2}} \leq \delta_i/2$. In particular, the triangle inequality implies that if $\overline{v}$ is the closest element of $\mathcal{M}_q(A) \cap \mathcal{B}$ to u, then $\|\overline{v} - u\|_{W^{1,2}} \leq \delta_i$. Thus, local quantitative stability implies, cf. Proposition 3.2,

$$\mathcal{Q}_q(u) - 1 \geq C(v_i) d_{\delta_i}(u, \mathcal{M}_q(A) \cap \mathcal{B})^{2+\gamma_i} \geq C_0 d(u, \mathcal{M}_q(A) \cap \mathcal{B})^{2+\gamma_0}.$$

On the other hand, in the case $d(u, \mathcal{M}_q(A) \cap \mathcal{B}) \geq \delta_0/4$, we argue as follows. By (the contrapositive of) Proposition 2.1, we get the existence of $\epsilon > 0$ such that

$$u \in \mathcal{B}, \quad d(u, \mathcal{M}_q(A) \cap \mathcal{B}) > \frac{\delta_0}{4} \quad \Rightarrow \quad \frac{1 - \mathrm{Vol}_g(M)^{2/q-1} \|u\|_{L^2}^2}{\|\nabla u\|_{L^2}^2} \leq A_q^{\mathrm{opt}}(M) - \epsilon,$$

(notice that, as we require $d(u, \mathcal{M}_q(A) \cap \mathcal{B}) > \delta_0/4$, it follows automatically that u cannot be a constant, hence the ratio is well defined). Rearranging, the last conclusion becomes

$$\mathcal{Q}_q(u) - 1 > \epsilon \|\nabla u\|_{L^2}^2.$$

We claim that, there exist $l > 0$ so that

$$u \in \mathcal{B}, \quad d(u, \mathcal{M}_q(A) \cap \mathcal{B}) > \frac{\delta_0}{4} \quad \Rightarrow \quad \|\nabla u\|_{L^2} > l.$$

If not, we could find a sequence $(u_n) \subseteq W^{1,2}(M)$ with $\|u_n\|_{L^q} = 1$ for all $n \in \mathbb{N}$ so that $d(u, \mathcal{M}_q(A) \cap \mathcal{B}) > \delta_0/4$ holds for all $n \in \mathbb{N}$ but $\|\nabla u_n\|_{L^2} \to 0$ as $n \uparrow \infty$. Up to a not relabelled subsequence, we thus infer that there is $u \in W^{1,2}(M)$ so that u_n converges strong in L^2 and weak in $W^{1,2}(M)$ to u. However, by lower semicontinuity, it follows that $\|\nabla u\|_{L^2} = 0$. In particular, u must be constant by the Poincaré inequality, say $u \equiv c$ for some $c \in \mathbb{R} \setminus \{0\}$ and therefore we get directly $\|u_n\|_{W^{1,2}} \to \|u\|_{W^{1,2}}$ as $n \uparrow \infty$. Since $W^{1,2}$ is Hilbert and u_n is already $W^{1,2}$-weak convergent, there is therefore strong $W^{1,2}$-convergence to the constant c. In particular, the Sobolev inequality implies that $u_n \to c$ in L^q (note that this is not trivial when $q = 2^*$). In particular, $\|c\|_{L^q} = 1$ and therefore c is a competitor in $\mathcal{M}_q(A) \cap \mathcal{B}$. However, this would violate the uniform condition

$$\frac{\delta_0}{4} < d(u_n, \mathcal{M}_q(A) \cap \mathcal{B}) \leq \frac{\|u_n - c\|_{W^{1,2}}}{\|u_n\|_{W^{1,2}}},$$

for n large enough. Finally, setting $C = \min\{C_0, l^2 \epsilon\}$ we conclude the proof of Theorem 1.1 (note that $d(u, \mathcal{M}_q(A)) \leq 1$ always hold).

Let us now discuss the straightforward modifications of the above arguments to obtain the proof of Theorem 1.2. In this case, we simply need to consider $q = 2^*$, $A = \mathsf{S}_d^2$, $B = B_{2^*}^{\mathrm{opt}}(M)$ and $\mathcal{M}^1 = \mathcal{M}_{2^*}(B) \cap \mathcal{B}$. When $d(u, \mathcal{M}_{2^*}(B) \cap \mathcal{B}) \leq \delta_0/4$ we repeat *verbatim* the above arguments, the only difference being the use of Theorem 2.3 and (2.3) replacing the role of Proposition 2.1. Finally, the case $d(u, \mathcal{M}_{2^*}(B) \cap \mathcal{B}) \geq \delta_0/4$ is now excluded by the last assumption in Theorem 1.2 choosing δ accordingly. $\square$

Remark 3.3. As already discussed in the Introduction, the concentration-compactness result in Proposition 2.1 made it possible to deduce in Theorem 1.1 that, when u is far away from the set of normalized optimizers, $\mathcal{Q}_q(u)$ is far away from its minimum value. This principle does not appear in [DD01] (recall Theorem 2.3 and (2.3)). Consequently, the only information we get when applying our strategy in Theorem 1.2 is a stability for functions that are *assumed* sufficiently close to the optimizers.

4. Degenerate stability

In this part, we prove degenerate stability results for $A_q^{\mathrm{opt}}(M)$.

4.1. **Proof of Theorem 1.3.** Denote by $\lambda(M)$ the first non-zero eigenvalue of the Laplace-Beltrami operator on M, and let $\varphi \in W^{1,2}(M)$ be the corresponding eigenfunction satisfying

$$(4.1) \qquad -\Delta\varphi = \lambda(M)\varphi, \qquad \int \varphi \, d\mathrm{Vol}_g = 0.$$

Recall that $\lambda(M)$ admits the following variational characterization

$$\lambda(M) = \inf_{\substack{f \in W^{1,2}(M) \\ f \neq cst.}} \frac{\|\nabla f\|_{L^2}^2}{\|f - \bar{f}\|_{L^2}^2},$$

where $\bar{f} = \int f \, d\mathrm{Vol}_g$. Under the standing hypotheses, we claim that

$$(4.2) \qquad \frac{q-2}{\lambda(M)} \mathrm{Vol}_g(M)^{\frac{2}{q}-1} = A_q^{\mathrm{opt}}(M).$$

That the left-hand side is bounded by the right-hand is follows from the well-known principle that the Sobolev inequality implies the Poincaré inequality on M, see e.g. the argument in [BGL14, Proposition 6.2.2]. For the converse inequality, we argue as follows. By definition of $A_q^{\mathrm{opt}}(M)$, let us consider a sequence $(u_n) \subseteq W^{1,2}(M)$ of non-constant functions so that

$$\frac{\|u_n\|_{L^q} - \mathrm{Vol}_g(M)^{2/q-1}\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \to A_q^{\mathrm{opt}}(M), \qquad \text{as } n \uparrow \infty.$$

Without loss of generality, we can suppose that $\|u_n\|_{L^q} = 1$ for all $n \in \mathbb{N}$. We are in a position to invoke Proposition 2.1 to deduce that, up to a non-relabelled subsequence, $u_n \to u$ in L^q and in $W^{1,2}$, for some non-zero $u \in \mathcal{M}_q(A)$. By assumption, we must have that $u \equiv c$ for $c = \mathrm{Vol}_g(M)^{-1/q}$ (as constant functions are the only extremal functions and such c has unitary L^q-norm). By the quantitative linearization principle for Sobolev inequalities [NV22, Lemma 6.7] (and by suitably scaling the volume in the norms, as there it is assumed a probability reference measure), setting $f_n := u_n - c$, we deduce

$$A_q^{\mathrm{opt}}(M) = \lim_{n \uparrow \infty} \frac{\|u_n\|_{L^q} - \mathrm{Vol}_g(M)^{2/q-1}\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}}$$

$$= (q-2)\mathrm{Vol}_g(M)^{\frac{2}{q}-1} \lim_{n \uparrow \infty} \frac{\int |f_n - \bar{f}_n|^2 \, d\mathrm{Vol}_g}{\|\nabla f_n\|_{L^2}} \leq \frac{q-2}{\lambda(M)} \mathrm{Vol}_g(M)^{\frac{2}{q}-1}.$$

This completes the proof of (4.2).

Let us now consider the constant extremal $u = c = \mathrm{Vol}_g(M)^{-1/q}$. By scaling, we can assume that the eigenfunction φ in (4.1) satisfies $\|\varphi\|_{L^q} = 1$. Then, the condition $\int \varphi \, d\mathrm{Vol}_g = 0$ implies

that $\varphi \in T_u \mathcal{B}$. Hence, we can Taylor expand the functional $\mathcal{Q}_q$ around $u = c$, in the direction φ to infer

$$\mathcal{Q}_q(c + \epsilon\varphi) - 1 = \mathcal{Q}_q(c + \epsilon\varphi) - \mathcal{Q}_q(c) = \epsilon\nabla_\mathcal{B}\mathcal{Q}_q(c)[\varphi] + \frac{\epsilon^2}{2}\nabla_\mathcal{B}^2\mathcal{Q}_q(\xi)[\varphi, \varphi]$$

$$= \epsilon^2 \frac{1}{2}\nabla_\mathcal{B}^2\mathcal{Q}_q(c)[\varphi, \varphi] + o(1)\|\epsilon\varphi\|_{W^{1,2}}^2,$$

where $\xi \in \mathcal{B}$ is a geodesic point between c and $c + \epsilon\varphi$, and $o(1)$ goes to zero as $\epsilon \downarrow 0$. By the computation of Lemma 2.4, we know that

$$\frac{1}{2}\nabla_\mathcal{B}^2\mathcal{Q}_q(c)[\varphi, \varphi] = A_q^{\mathrm{opt}}(M)\int \lambda(M)\varphi^2 \, d\mathrm{Vol}_g + \mathrm{Vol}_g(M)^{\frac{2}{q}-1}\int \varphi^2 \, d\mathrm{Vol}_g$$

$$- (q-1)\mathcal{Q}_q(c)\int c^{q-2}\varphi^2 \, d\mathrm{Vol}_g$$

$$= (q-1)\mathrm{Vol}_g(M)^{\frac{2}{q}-1}\int \varphi^2 \, d\mathrm{Vol}_g - \mathrm{Vol}_g(M)^{\frac{2}{q}-1}(q-1)\int \varphi^2 \, d\mathrm{Vol}_g = 0,$$

having used (4.2), and the choice $c = \mathrm{Vol}_g(M)^{-1/q}$. Finally, by contradiction, if $\gamma = 0$ were possible, we would reach

$$C \inf_{v \in \mathcal{M}_q(A)} \|c + \epsilon\varphi - v\|_{W^{1,2}}^2 \le \mathcal{Q}_q(c + \epsilon\varphi) - 1 \approx \|\epsilon\varphi\|_{W^{1,2}}^2 \cdot o(1)$$

which is false for $\epsilon \ll 1$. This concludes the proof. $\qquad\square$

4.2. **Proof of Corollary 1.4.** We prove first i). Let $d > 2$ and consider any $q \in (2, 2^*)$. Recall [BE85, BVV91, Bec93] (also rearranging in [Fra22, Eq. (5)]) that

$$A_q^{\mathrm{opt}}(\mathbb{S}^d) = \frac{q-2}{d}\mathrm{Vol}(\mathbb{S}^d)^{2/q-1},$$

and that the only extremal functions are constant functions. Thus, Theorem 1.3 guarantees that the stability is degenerate. To check that $\gamma = 2$ is optimal, we can rearrange the quantitative stability in [Fra22, Theorem 2] and we have for some $c_{d,q} > 0$

$$\frac{A_q^{\mathrm{opt}}(\mathbb{S}^d)\|\nabla u\|_{L^2}^2 - \mathrm{Vol}(\mathbb{S}^d)^{2/q-1}\|u\|_{L^2}^2}{\|u\|_{L^q}^2} - 1 \ge c_{d,q}\left(\inf_{c\in\mathbb{R}} \frac{\|u - c\|_{W^{1,2}}}{\|u\|_{W^{1,2}}}\right)^4,$$

for every $u \in W^{1,2}(\mathbb{S}^d) \setminus \{0\}$. Since in [Fra22, Theorem 2] the exponent 4 is proven to be sharp, we directly deduce that in $\mathbb{S}^d$, and in the range $q \in (2, 2^*)$, Theorem 1.1 holds sharp with $\gamma = 2$.

We conclude by proving ii). For $d > 2$, consider the closed Riemannian manifold

(4.3) $$M_* := \mathbb{S}^1\left(\frac{1}{\sqrt{d-2}}\right) \times \mathbb{S}^{d-1},$$

equipped with the product metric denoted by g. In [Fra22], the author consider the following *conformally invariant Sobolev inequality*:

(4.4) $$Y(M_*)\|u\|_{L^{2^*}}^2 \le \mathcal{E}_g(u), \qquad \forall u \in W^{1,2}(M_*)$$

where $\mathcal{E}_g(u) := \int |\nabla u|^2 + \frac{d-2}{4(d-1)}\mathrm{R}_g u^2 \, d\mathrm{Vol}_g$, and $Y(M_*)$ is the *Yamabe constant* of M_*. Notice

$$\mathrm{R}_g = (d-2)(d-1), \qquad \mathrm{Vol}_g(M_*) = \frac{2\pi}{\sqrt{d-2}}\mathrm{Vol}(\mathbb{S}^{d-1}).$$

Moreover, the Yamabe constant can be also computed explicitly (see [Fra22] and [Sch89])

$$Y(M_*) = \frac{(d-2)^2}{4}\mathrm{Vol}_g(M_*)^{2/d}.$$

Hence, rearranging in (4.4), we have for all $u \in W^{1,2}(M_*)$

$$(4.5) \qquad \|u\|_{L^{2^*}}^2 \leq \frac{4}{(d-2)^2}\mathrm{Vol}_g(M_*)^{-2/d}\|\nabla u\|_{L^2}^2 + \mathrm{Vol}_g(M_*)^{-2/d}\|u\|_{L^2}^2.$$

Since (4.4) is sharp by definition of the Yamabe constant $Y(M_*)$, we deduce that

$$A_{2^*}^{\mathrm{opt}}(M_*) = \frac{4}{(d-2)^2}\mathrm{Vol}_g(M_*)^{-2/d}.$$

Now, constant functions are the only extremal functions in (4.4) and hence also in (4.5). Thus, to apply Theorem 1.3 we need to check that

$$(4.6) \qquad \mathsf{S}_d^2 < A_{2^*}^{\mathrm{opt}}(M_*),$$

Recalling that $\mathsf{S}_d^2 = \frac{4}{d(d-2)}\mathrm{Vol}(\mathbb{S}^d)^{-2/d}$ and using the above computations, (4.6) becomes

$$\frac{4}{d(d-2)} < \frac{4}{(d-2)^2}(d-2)^{1/d}.$$

which is always true for $d > 2$. Thus, Theorem 1.3 guarantees that the stability is degenerate. We now check that $\gamma = 2$ is optimal and conclude. By [Fra22, Theorem 3], there is a constant $c_d > 0$ so that for every $u \in W^{1,2}(M_*) \setminus \{0\}$ it holds sharp

$$(4.7) \qquad \mathcal{E}_g(u) - Y(M_*)\|u\|_{L^{2^*}}^2 \geq c_d \inf_{c \in \mathbb{R}} \frac{\mathcal{E}_g(u-c)^2}{\mathcal{E}_g(u)}.$$

Notice that $\mathcal{E}_g(u)$ is comparable up to dimensional constants to $\|u\|_{W^{1,2}}^2$ (thanks to the fact that R_g is constant). Hence, we can rearrange in (4.7) using (4.5) to deduce

$$\frac{A_{2^*}^{\mathrm{opt}}(M_*)\|\nabla u\|_{L^2}^2 + \mathrm{Vol}(M_*)^{-2/d}\|u\|_{L^2}^2}{\|u\|_{L^{2^*}}} - 1 \geq c_d \left(\inf_{c \in \mathbb{R}} \frac{\|u-c\|_{W^{1,2}}}{\|u\|_{W^{1,2}}} \right)^4,$$

for every $u \in W^{1,2}(M_*) \setminus \{0\}$, and for a possibly different constant c_d. Finally, this concludes the proof of ii) since the optimality in (4.7) implies that $\gamma = 2$ is sharp. $\square$

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